Finally, simplest periodic motion of constrained systems can be investigated when the spheres move in some (e. g. homogeneous) field of force with the result that conditions (2. 1) no longer hold.

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ON THE MOTION EQUATIONS OF NONHOLONOMIC MECHANICAL SYSTEMS IN POINCARÉ-CHETAEV VARIABLES

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The Poincaré-Chetaev equations for holonomic mechanical systems have been written by Poincaré [1] and generalized by Chetaev to the dependent variables case [2]. The purpose of the present paper is to extend the mentioned method to the case of nonholonomic systems.

1. Formulation of the problem. Let us consider a nonholonomic mechanical system defined by the n Poincaré-Chetaev variables $x_1, \ldots, x_n$ [2], which are subject, in real displacements, to the following p holonomic and q nonholonomic constraints

$$a_{s1}x_1' + \ldots + a_{sn}x_n' + a_s = 0 \qquad (s = 1, \ldots, p) \tag{1.1}$$

$$\alpha_{\nu1}x_1' + \ldots + \alpha_{\nu n}x_n' + \alpha_\nu = 0 \qquad (\nu = 1, \ldots, q) \tag{1.2}$$

and in possible displacements, to Eqs. [3]

$$a_{s1}\delta x_1 + \ldots + a_{sn}\delta x_n = 0 \qquad (s = 1, \ldots, p) \tag{1.3}$$

$$\alpha_{\nu1}\delta x_1 + \ldots + \alpha_{\nu n}\delta x_n = 0 \qquad (\nu = 1, \ldots, q) \tag{1.4}$$

Here a_{si}, a_s, $\alpha_{\nu i}$, α_ν are functions of the time t and the variables $x_1, \ldots, x_n$; x_i' and δx_i are the derivatives and variations of the variables x_i . The constraints (1. 1) and (1, 3) form a completely integrable system of p Pfaffian forms [4]. The constraints (1, 2) are not integrable, and may not mutually form completely integrable systems, nor with respect to (1, 1).

Let all the constraints be ideal, and let the forces have a force function. Let us write the equations of motion for this nonholonomic system by the Poincaré-Chetaev method [1 and 2].

2. Construction of infinitesimal displacement operators. As is known, a closed system of displacement operators is constructed in the Poincaré-Chetaev method [2]. We find these operators for a given system as in [2], by using the holonomic constraints (1, 1) for the actual displacements, and (1, 3) for the possible displacements.

Hence, let $\omega_1, \ldots, \omega_k$ be the parameters of the possible displacements, and $\eta_1, \ldots, \eta_k$ the parameters of the real displacements. The corresponding operators will be

$$X_0 = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \xi_0{}^i \frac{\partial}{\partial x_i}, \qquad X_s = \sum_{i=1}^{n} \xi_s{}^i \frac{\partial}{\partial x_i} \qquad (s = 1, \ldots, k;\ k = n - p) \quad (2.1)$$

Here ξ_0^i, ξ_s^i are functions of the variables and the time.

Then changes in the arbitrary function of the position of the mechanical system $f(t, x_1, \ldots, x_n)$ in the possible and real displacements admitted by (1, 1) and (1, 3) will be by definition [2]

$$df = dt\Big[X_0(f) + \sum_{s=1}^{k} \eta_s X_s(f)\Big], \qquad \delta f = \sum_{s=1}^{k} \omega_s X_s(f) \quad (2.2)$$

These operators X_0 and $X_1, \ldots, X_k$ satisfy the relationships

$$(X_0, X_\alpha) = \sum_{\beta=1}^{k} C_{0\alpha\beta} X_\beta, \qquad (X_s, X_\alpha) = \sum_{\beta=1}^{k} C_{s\alpha\beta} X_\beta \qquad (s, \alpha = 1, \ldots, k) \quad (2.3)$$

Here $C_{0\alpha\beta}$ and $C_{s\alpha\beta}$ are functions of $x_1, \ldots, x_n$ and t dependent on the selection of the set of displacement parameters.

3. Equations of motion. Let us define x_i' and δx_i according to (2, 2) for the function $f = x_i$ $(i = 1, \ldots, n)$ and let us substitute them into (1, 1), (1, 3) and (1, 2), (1, 4). Then the constraints (1, 1) and (1, 3) transform into an identity, and the constraints (1, 2) and (1, 4) become

$$\eta_\nu = \sum_{s=1}^{l} c_{\nu s} \eta_s + c_\nu, \qquad \omega_\nu = \sum_{s=1}^{l} c_{\nu s} \omega_s \qquad (\nu = l + 1, \ldots, k) \quad (3.1)$$

after we cut off the last (from k) displacement parameters relative to q, which we assume to be dependent because of the nonholonomic constraints (1, 2) and (1, 4).

Here $l = k - q$ is the number of independent displacement parameters: $c_{\nu i}$ and c_ν are functions of the variables $x_1, \ldots, x_n$ and the time t.

The general dynamics equation may be reduced to (*)

$$\sum_{\alpha=1}^{k} \omega_\alpha \Big[\frac{d}{dt}\frac{\partial T}{\partial \eta_\alpha} - \sum_{\beta=1}^{k} C_{0\alpha\beta}\frac{\partial T}{\partial \eta_\beta} - \sum_{s=1}^{k} \eta_s \sum_{\beta=1}^{k} C_{s\alpha\beta}\frac{\partial T}{\partial \eta_\beta} - X_\alpha(T + U)\Big] = 0 \quad (3.2)$$

by utilizing (2, 2) and (2, 3).

Here $T = T(t, x_1, \ldots, x_n, \eta_1, \ldots, \eta_k)$ is the kinetic energy; $U = U(t, x_1, \ldots, x_n)$ the force function of the system.

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*) See K. E. Shurova, Some properties of the Poincaré equations. Dissertation, Moscow State Univ., 1958.

If the $\omega_1, \ldots, \omega_k$ are independent, i. e. if the system is subject only to the holonomic constraints (1, 1), then we obtain the Poincaré-Chetaev Eq. from (3, 2)

$$\frac{d}{dt}\frac{\partial T}{\partial \eta_\alpha} - \sum_{\beta=1}^{k} C_{0\alpha\beta}\frac{\partial T}{\partial \eta_\beta} - \sum_{s=1}^{k}\eta_s\sum_{\beta=1}^{k} C_{s\alpha\beta}\frac{\partial T}{\partial \eta_\beta} - X_\alpha(T+U) = 0 \qquad (\alpha = 1, \ldots, k) \tag{3.3}$$

When the nonholonomic constraints (1, 2) transformed to the form (3, 1), are taken into account, Eqs. (3, 3) do not hold. To obtain the equations of motion in this case, following Chaplygin [5], we replace all the dependent possible-displacement parameters ω_ν $(\nu = l+1, \ldots, k)$ in (3, 2) by means of (3, 1). Then, because of the independence of the $\omega_1, \ldots, \omega_l$, we obtain

$$\frac{d}{dt}\frac{\partial T}{\partial \eta_\alpha} - \sum_{\beta=1}^{k} C_{0\alpha\beta}\frac{\partial T}{\partial \eta_\beta} - \sum_{s=1}^{k}\eta_s\sum_{\beta=1}^{k} C_{s\alpha\beta}\frac{\partial T}{\partial \eta_\beta} - X_\alpha(T+U) +$$
$$+ \sum_{\nu=l+1}^{k} c_{\nu\alpha}\left[\frac{d}{dt}\frac{\partial T}{\partial \eta_\nu} - \sum_{\beta=1}^{k} C_{0\nu\beta}\frac{\partial T}{\partial \eta_\beta} - \sum_{s=1}^{k}\eta_s\sum_{\beta=1}^{k} C_{s\nu\beta}\frac{\partial T}{\partial \eta_\beta} - X_\nu(T+U)\right] = 0$$
$$(\alpha = 1, \ldots, l) \tag{3.4}$$

These Eqs. may be transformed to a form which does not contain the dependent parameters of the real displacements η_ν $(\nu = l+1, \ldots, k)$. To do this, we separate all the sums in (3, 4) into separate sums from 1 to ℓ and from $\ell+1$ to k, we replace all the dependent parameters η_ν in them by means of (3, 1) and we obtain

$$\frac{d}{dt}\frac{\partial T}{\partial \eta_\alpha} + \sum_{\nu=l+}^{k} c_{\nu\alpha}\frac{d}{dt}\frac{\partial T}{\partial \eta_\nu} - \sum_{\beta=1}^{l} k_{0\alpha\beta}\frac{\partial T}{\partial \eta_\beta} - \sum_{s=1}^{l}\eta_s\sum_{\beta=1}^{l} k_{s\alpha\beta}\frac{\partial T}{\partial \eta_\beta} -$$
$$- \sum_{\nu=l+1}^{k} k_{0\alpha\nu}\frac{\partial T}{\partial \eta_\nu} - \sum_{s=1}^{l}\eta_s\sum_{\nu=l+1}^{k} k_{s\alpha\nu}\frac{\partial T}{\partial \eta_\nu} - Y_\alpha(T+U) = 0 \qquad (\alpha = 1, \ldots, l) \tag{3.5}$$

Here

$$Y_\alpha = X_\alpha + \sum_{\nu=l+1}^{k} c_{\nu\alpha}X_\nu \qquad (\alpha = 1, \ldots, l) \tag{3.6}$$

$$k_{0\alpha\beta} = C_{0\alpha\beta} + \sum_{\nu=l+1}^{k} c_{\nu\alpha}C_{0\nu\beta} + \sum_{\mu=l+1}^{k} c_\mu\left(C_{\mu\alpha\beta} + \sum_{\nu=l+1}^{k} c_{\nu\alpha}C_{\mu\nu\beta}\right)$$
$$k_{s\alpha\beta} = C_{s\alpha\beta} + \sum_{\nu=l+1}^{k} c_{\nu\alpha}C_{s\nu\beta} + \sum_{\mu=l+1}^{k} c_{\mu s}\left(C_{\mu\alpha\beta} + \sum_{\nu=l+1}^{k} c_{\nu\alpha}C_{\mu\nu\beta}\right)$$
$$(s, \alpha = 1, \ldots, l;\ \beta = 1, \ldots, l, l+1, \ldots, k) \tag{3.7}$$

Let Θ denote the function obtained from T by replacing all the dependent real-displacement parameters $\eta_{l+1}, \ldots, \eta_k$ by means of (3, 1)

$$\Theta(t, x_1, \ldots, x_n, \eta_1, \ldots, \eta_l) = T(t, x_1, \ldots, x_n, \eta_1, \ldots, \eta_l, \eta_{l+1}, \ldots, \eta_k) \tag{3.8}$$

Then we have the following relationships for T and Θ

$$Y_\alpha(T) = Y_\alpha(\Theta) - \sum_{\nu=l+1}^{k}\frac{\partial T}{\partial \eta_\nu}Y_\alpha(c_\nu) - \sum_{s=1}^{l}\eta_s\sum_{\nu=l+1}^{k}\frac{\partial T}{\partial \eta_\nu}Y_\alpha(c_{\nu\alpha}) \qquad (\alpha = 1, \ldots, l) \tag{3.9}$$

$$\frac{\partial T}{\partial \eta_\alpha} = \frac{\partial \Theta}{\partial \eta_\alpha} - \sum_{\nu=l+1}^{k}\frac{\partial T}{\partial \eta_\nu}c_{\nu\alpha} \qquad (\alpha = 1, \ldots, l) \tag{3.10}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \eta_\alpha} = \frac{d}{dt}\frac{\partial \Theta}{\partial \eta_\alpha} - \sum_{\nu=l+1}^{k} c_{\nu\alpha}\frac{d}{dt}\frac{\partial T}{\partial \eta_\nu} - \sum_{\nu=l+1}^{k}\frac{\partial T}{\partial \eta_\nu}\frac{dc_{\nu\alpha}}{dt} \qquad (\alpha = 1, \ldots, l) \quad (3.11)$$

The derivatives $dc_{\nu\alpha}/dt$ in (3.11) may be found by means of (2.2), which yield by virtue of (3.1)

$$\frac{dc_{\nu\alpha}}{dt} = Y_0(c_{\nu\alpha}) + \sum_{s=1}^{l}\eta_s Y_s(c_{\nu\alpha}) \qquad (\alpha = 1, \ldots, l;\ \nu = l+1, \ldots, k) \quad (3.12)$$

Here

$$Y_0 = X_0 + \sum_{\nu=l+1}^{k} c_\nu X_\nu \qquad (3.13)$$

Substituting (3.9), (3.10) and (3.11), taking account of (3.12) into (3.5), we obtain

$$\frac{d}{dt}\frac{\partial \Theta}{\partial \eta_\alpha} - \sum_{\beta=1}^{l} k_{0\alpha\beta}\frac{\partial \Theta}{\partial \eta_\beta} - \sum_{s=1}^{l}\eta_s\sum_{\beta=1}^{l} k_{s\alpha\beta}\frac{\partial \Theta}{\partial \eta_\beta} - \sum_{\nu=l+1}^{k}\frac{\partial T}{\partial \eta_\nu}\Big[k_{0\alpha\nu} - \sum_{\beta=1}^{l} c_{\nu\beta}k_{0\alpha\beta} +$$

$$+ Y_0(c_{\nu\alpha}) - Y_\alpha(c_\nu)\Big] - \sum_{s=1}^{l}\eta_s\sum_{\nu=l+1}^{k}\frac{\partial T}{\partial \eta_\nu}\Big[k_{s\alpha\nu} - \sum_{\beta=1}^{l} c_{\nu\beta}k_{s\alpha\beta} + Y_s(c_{\nu\alpha}) - Y_\alpha(c_{\nu s})\Big] -$$

$$- Y_\alpha(\Theta + U) = 0 \qquad (\alpha = 1, \ldots, l) \quad (3.14)$$

This is the equation of motion of nonholonomic systems in Poincaré-Chetaev variables. Together with n equations obtained from (2.2) with (3.1) for the function $f = x_i$ $(i = 1, \ldots, n)$,

$$\frac{dx_i}{dt} = \xi_0{}^i + \sum_{\nu=l+1}^{k} c_\nu\xi_\nu{}^i + \sum_{\alpha=1}^{l}\eta_\alpha\Big(\xi_\alpha{}^i + \sum_{\nu=l+1}^{k} c_{\nu\alpha}\xi_\nu{}^i\Big) \qquad (i = 1, \ldots, n) \quad (3.15)$$

they yield $n+l$ first order equations for the determination of $x_1, \ldots, x_n$ and $\eta_1, \ldots, \eta_l$ as a function of time t.

4. Particular cases. Let us show that (3.14) contains, as particular cases, the Chaplygin Eqs. [5] and the Volterra-Voronets Eq. [8 and 9] for nonholonomic systems.

In [5] Chaplygin examined a nonholonomic system defined by the generalized coordinates $x_1, \ldots, x_n$ subject to $n-l$ nonholonomic constraints (l is the number of independent velocities)

$$x_\nu' = c_{\nu 1}x_1' + \ldots + c_{\nu l}x_l' \qquad (\nu = l+1, \ldots, n) \quad (4.1)$$

Here $c_{\nu s}$ are functions independent of time t and of $x_{l+1}, \ldots, x_n$, which are cyclic coordinates of the mechanical system; x_i' are derivatives of the variables x_i.

He obtained the equations of motion in the form

$$\frac{d}{dt}\frac{\partial \Theta}{\partial x_\alpha'} - \sum_{s=1}^{l} x_s'\sum_{\nu=l+1}^{n}\frac{\partial T}{\partial x_\nu'}\left(\frac{\partial c_{\nu\alpha}}{\partial x_s} - \frac{\partial c_{\nu s}}{\partial x_\alpha}\right) - \frac{\partial(\Theta + U)}{\partial x_\alpha} = 0 \quad (4.2)$$

$$(\alpha = 1, \ldots, l)$$

These Chaplygin Eqs. may be obtained from the Poincaré-Chetaev Eqs. (3.14). To do this, we take $x_1, \ldots, x_n$ as Poincaré-Chetaev variables. Then no holonomic constraints of (1.1) type exist between these variables, they are only subject to the nonholonomic constraints (1.2) in the form (1.4).

Let us take $x_1', \ldots, x_n'$ as the real-displacement parameters $\eta_1, \ldots, \eta_n$ and we take $\delta x_1, \ldots, \delta x_n$ as the possible-displacement parameters $\omega_1, \ldots, \omega_n$. In this case the displacement operators (2.1) will be

$$X_0 = \frac{\partial}{\partial t}, \qquad X_s = \frac{\partial}{\partial x_s} \qquad (s = 1, \ldots, n) \tag{4.3}$$

These operators commute, hence all the quantities $C_{0\alpha\beta}$, $C_{s\alpha\beta}$ in (2.3) and $k_{0\alpha\beta}$, $k_{s\alpha\beta}$ in (3.7) equal zero; all the terms containing $k_{0\alpha\beta}$, $k_{s\alpha\beta}$ are missing from (3.14).

The equations of the nonholonomic constraints (4.1) take the form (3.1)

$$\eta_\nu = \sum_{\alpha=1}^{l} c_{\nu\alpha}\eta_\alpha, \qquad \omega_\nu = \sum_{\alpha=1}^{l} c_{\nu\alpha}\omega_\alpha \qquad (\nu = l+1, \ldots, n) \tag{4.4}$$

The operators (3.13) and (3.16) will be

$$Y_0 = \frac{\partial}{\partial t}, \qquad Y_\alpha = \frac{\partial}{\partial x_\alpha} + \sum_{\nu=l+1}^{n} c_{\nu\alpha}\frac{\partial}{\partial x_\nu} \qquad (\alpha = 1, \ldots, l) \tag{4.5}$$

Because $c_\nu = 0$ and $c_{\nu\alpha}$, Θ and U are independent of time and the cyclic displacements $x_{l+1}, \ldots, x_n$, these latter yield

$$Y_0(c_{\nu\alpha}) - Y_\alpha(c_\nu) = 0, \quad Y_s(c_{\nu\alpha}) - Y_\alpha(c_{\nu s}) = \frac{\partial c_{\nu\alpha}}{\partial x_s} - \frac{\partial c_{\nu s}}{\partial x_\alpha}, \quad Y_\alpha(\Theta + U) = \frac{\partial(\Theta + U)}{\partial x_\alpha}$$
$$(\alpha, s = 1, \ldots, l;\ \nu = l+1, \ldots, n) \tag{4.6}$$

Hence, substituting (4.6) into (3.14), and first replacing η_α, η_ν by x_α', x_ν', we obtain the Chaplygin Eqs. (4.2).

Also the generalization of the mentioned equations in Poincaré-Chetaev variables may be obtained from (3.14).

Let

1°. All the $k-l$ displacement operators $X_{l+1}, \ldots, X_k$ in (2.1), which correspond to the dependent displacement parameters η_ν and ω_ν from (3.1), be cyclic displacements according to Chetaev [2], and let X_0 commute with all X_ν, i.e. let the following conditions be satisfied:

$$(X_\alpha, X_\nu) = 0,\ X_\nu(T + U) = 0,\ (X_0, X_\nu) = 0\ (\alpha = 1, \ldots, l,\ l+1, \ldots, k;\ \nu = l+1, \ldots, k) \tag{4.7}$$

2°. For the nonholonomic constraints reduced to the form (3.1), there are the relationships

$$X_\mu(c_{\nu\alpha}) = 0, \quad X_\mu(c_\nu) = 0 \quad (\alpha = 1, \ldots, l;\ \nu, \mu = l+1, \ldots, k) \tag{4.8}$$

Then (3.14) become

$$\frac{d}{dt}\frac{\partial\Theta}{\partial\eta_\alpha} - \sum_{\beta=1}^{l} C_{0\alpha\beta}\frac{\partial\Theta}{\partial\eta_\beta} - \sum_{s=1}^{l}\eta_s\sum_{\beta=1}^{l} C_{s\alpha\beta}\frac{\partial\Theta}{\partial\eta_\beta} -$$
$$- \sum_{\nu=l+1}^{l}\frac{\partial T}{\partial\eta_\nu}\left[C_{0\alpha\nu} - \sum_{\beta=1}^{l} c_{\nu\beta}C_{0\alpha\beta} + X_0(c_{\nu\alpha}) - X_\alpha(c_\nu)\right] -$$
$$- \sum_{s=1}^{l}\eta_s\sum_{\nu=l+1}^{k}\frac{\partial T}{\partial\eta_\nu}\left[C_{s\alpha\nu} - \sum_{\beta=1}^{l} c_{\nu\beta}C_{s\alpha\beta} + X_s(c_{\nu\alpha}) - X_\alpha(c_{\nu s})\right] - X_\alpha(\Theta + U) = 0$$
$$(\alpha = 1, \ldots, l) \tag{4.9}$$

This is the generalized Chaplygin Eq. in Poincaré-Chetaev variables.

When the variables $x_1, \ldots, x_n$ are generalized coordinates, and the constraints (3.1), i.e. (4.4) are independent of time, and $c_\nu = 0$, then (4.9) take the form of the Chaplygin Eqs. (4.2).

In exactly the same manner it can be shown that Eqs. (3.14) contain the Voronets Eqs. [6] and their general form, the generalized Chaplygin-Voronets Eqs. [7] for nonholonomic systems in generalized coordinates (*).

The Volterra Eqs. for nonholonomic systems in nonholonomic coordinates were obtained in 1897 in [8], and Voronets obtained their generalization in 1903 in [9]. In the mentioned paper (Chapter 3) Voronets considered a mechanical system defined by generalized coordinates $x_1, \ldots, x_n$ subject to $n-l$ nonholonomic constraints (l is the number of independent velocities), which express the derivatives $x_1', \ldots, x_n'$ in terms of l independent quantities φ_s' which are functions o time

$$x_i' = c_{i1}\varphi_1' + \ldots + c_{il}\varphi_l' + c_i \quad (i = 1, \ldots, n) \tag{4.10}$$

Here c_{is}, c_i are functions of time and the coordinates. For this system Voronets obtained equations of motion in the form (4.11)

$$\frac{d}{dt}\frac{\partial\Theta}{\partial\varphi_\alpha'} - \sum_{\beta=1}^{l} K_{\alpha\beta}\frac{\partial\Theta}{\partial\varphi_\beta'} - \sum_{\nu=l+1}^{n} L_{\alpha\nu}\frac{\partial T}{\partial x_\nu'} - \sum_{i=1}^{n} c_{i\alpha}\frac{\partial(\Theta + U)}{\partial x_i} = 0 \qquad (\alpha = 1, \ldots, l)$$

$$K_{\alpha\beta} = \sum_{j=1}^{l} b_{\beta j}\left(\frac{dc_{j\alpha}}{dt} - \sum_{i=1}^{n} c_{i\alpha}\frac{\partial x_j'}{\partial x_i}\right)$$
$$L_{\alpha\nu} = \frac{dc_{\nu\alpha}}{dt} - \sum_{i=1}^{n} c_{i\alpha}\frac{\partial x_\nu'}{\partial x_i} - \sum_{\beta=1}^{l} c_{\nu\beta}K_{\alpha\beta} \qquad \left(\begin{matrix}\alpha, \beta = 1, \ldots, l\\ \nu = l+1, \ldots, n\end{matrix}\right) \tag{4.12}$$

Here the quantities $b_{\beta j}$ are defined from the relationships

$$b_{\beta 1}c_{1\alpha} + \ldots + b_{\beta l}c_{l\alpha} = \delta_{\beta\alpha} \qquad (\alpha, \beta = 1, \ldots, l). \tag{4.13}$$

($\delta_{\beta\alpha}$ is the Kronecker delta).

Let us obtain (4.11) from (3.14). To do this, we take $x_1, \ldots, x_n$ as Poincaré-Chetaev variables. Then there will be no constraints of (1.1) type among the x_i; they are subject only to the nonholonomic constraints (1.2) in the form (4.10). Hence if $\varphi_1', \ldots, \varphi_l'$ and $x_{l+1}', \ldots, x_n'$ are taken as parameters of the real displacements $\eta_1, \ldots, \eta_l, \eta_{l+1}, \ldots, \eta_n$ then the displacement operators (2.1) and the quantities $C_{0\alpha\beta}, C_{s\alpha\beta}$ in (2.3) will be

$$X_0 = \frac{\partial}{\partial t} + \sum_{i=1}^{l} c_i\frac{\partial}{\partial x_i}, \quad X_s = \sum_{i=1}^{l} c_{is}\frac{\partial}{\partial x_i}, \quad X_\nu = \frac{\partial}{\partial x_\nu} \quad (s = 1, \ldots, l;\ \nu = l+1, \ldots, n) \tag{4.14}$$

$$C_{0\alpha\beta} = \sum_{j=1}^{l} b_{\beta j}[X_0(c_{j\alpha}) - X_\alpha(c_j)]; \qquad C_{0\nu\beta} = -\sum_{j=1}^{l} b_{\beta j}X_\nu(c_j)$$
$$C_{s\alpha\beta} = \sum_{j=1}^{l} b_{\beta j}[X_s(c_{j\alpha}) - X_\alpha(c_{js})]; \qquad C_{\alpha\nu\beta} = -\sum_{j=1}^{l} b_{\beta j}X_\nu(c_{j\alpha})$$

$$C_{0\alpha\mu} = C_{0\nu\mu} = C_{s\alpha\mu} = C_{\nu\alpha\mu} = C_{\nu\mu\beta} = C_{\nu\mu\gamma} = 0 \quad (s, \alpha, \beta = 1, \ldots, l;\ \nu, \mu, \gamma = l+1, \ldots, n) \tag{4.15}$$

Here the $b_{\beta j}$ are quantities determined from (4.13).

The equations of the nonholonomic constraints (3.1) from (4.10), the displacement

*) See also: M. I. Efimov. On Chaplygin equations for nonholonomic systems. Dissertation. Institute of Mechanics, Akad. Nauk SSSR, 1953.

operators (3, 13), (3, 6) will be the following:

$$\eta_\nu = \sum_{s=1}^{l} c_{\nu s}\eta_s + c_\nu, \qquad \omega_\nu = \sum_{s=1}^{l} c_{\nu s}\omega_s \qquad (\nu = l+1, \ldots, n) \tag{4.16}$$

$$Y_0 = \frac{\partial}{\partial t} + \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i}, \qquad Y_\alpha = \sum_{i=1}^{n} c_{i\alpha}\frac{\partial}{\partial x_i} \qquad (\alpha = 1, \ldots, l) \tag{4.17}$$

According to (3, 7) the quantities $k_{0\alpha\beta}$, $k_{s\alpha\beta}$ will be in this case

$$k_{0\alpha\beta} = \sum_{j=1}^{l} b_{\beta j}\left[Y_0(c_{j\alpha}) - Y_\alpha(c_j)\right], \qquad k_{0\alpha\nu} = k_{s\alpha\nu} = 0$$

$$k_{s\alpha\beta} = \sum_{j=1}^{l} b_{\beta j}\left[Y_s(c_{j\alpha}) - Y_\alpha(c_{js})\right] \qquad \begin{pmatrix} s, \alpha, \beta = 1, \ldots, l \\ \nu = l+1, \ldots, n \end{pmatrix} \tag{4.18}$$

Because of (4, 17) and (4, 18), Eqs. (3, 14) become

$$\frac{d}{dt}\frac{\partial\Theta}{\partial\eta_\alpha} - \sum_{\beta=1}^{l}\frac{\partial\Theta}{\partial\eta_\beta}\sum_{j=1}^{l} b_{\beta j}\left\{Y_0(c_{j\alpha}) - Y_\alpha(c_j) + \sum_{s=1}^{l}\eta_s\left[Y_s(c_{j\alpha}) - Y_\alpha(c_{js})\right]\right\} -$$

$$- \sum_{\nu=l+1}^{n}\frac{\partial T}{\partial\eta_\nu}\left\{-\sum_{\beta=1}^{l} c_{\nu\beta}\sum_{j=1}^{l} b_{\beta j}\left(Y_0(c_{j\alpha}) - Y_\alpha(c_j) + \sum_{s=1}^{l}\eta_s\left[Y_s(c_{j\alpha}) - Y_\alpha(c_{js})\right]\right) + \right.$$

$$\left. + Y_0(c_{\nu\alpha}) - Y_\alpha(c_\nu) + \sum_{s=1}^{l}\eta_s\left[Y_s(c_{\nu\alpha}) - Y_\alpha(c_{\nu s})\right]\right\} - \sum_{i=1}^{n} c_{i\alpha}\frac{\partial(\Theta + U)}{\partial x_i} = 0 \quad (\alpha = 1, \ldots, l) \tag{4.19}$$

After replacement of η_α by φ_α', η_ν by x_ν' and

$$Y_0(c_{j\alpha}) + \sum_{s=1}^{l}\eta_s Y_s(c_{j\alpha}) = \frac{dc_{j\alpha}}{dt}$$

$$Y_\alpha(c_j) + \sum_{s=1}^{l}\eta_s Y_\alpha(c_{js}) = Y_\alpha(x_j') = \sum_{i=1}^{n} c_{i\alpha}\frac{\partial x_j'}{\partial x_i}$$

these equations are reduced to

$$\frac{d}{dt}\frac{\partial\Theta}{\partial\varphi_\alpha'} - \sum_{\beta=1}^{l}\frac{\partial\Theta}{\partial\varphi_\beta'}\sum_{j=1}^{l} b_{\beta j}\left(\frac{dc_{j\alpha}}{dt} - \sum_{i=1}^{n} c_{i\alpha}\frac{\partial x_j'}{\partial x_i}\right) -$$

$$- \sum_{\nu=l+1}^{n}\frac{\partial T}{\partial x_\nu'}\left[\frac{dc_{\nu\alpha}}{dt} - \sum_{i=1}^{n} c_{i\alpha}\frac{\partial x_\nu'}{\partial x_i} - \sum_{\beta=1}^{l} c_{\nu\beta}\sum_{j=1}^{l} b_{\beta j}\left(\frac{dc_{j\alpha}}{dt} - \sum_{i=1}^{n} c_{i\alpha}\frac{\partial x_j'}{\partial x_i}\right)\right] -$$

$$- \sum_{i=1}^{n} c_{i\alpha}\frac{\partial(\Theta + U)}{\partial x_i} = 0 \qquad (\alpha = 1, \ldots, l) \tag{4.20}$$

In the notation (4, 12) these latter agree with the Voronets Eqs. (4, 11).

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